REGULARITY AT THE BOUNDARY AND TANGENTIAL REGULARITY

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ABSTRACT. For a pseudoconvex domain $D \subset \mathbb{C}^n$, we prove the equivalence of the local hypoellipticity of the system $(\bar{\partial}, \bar{\partial}^*)$ with the system $(\bar{\partial}_b, \bar{\partial}_b^*)$ induced in the boundary. This develops our former result in [5] which used the theory of the "harmonic" extension by Kohn. This technique is inadequate for the purpose of the present paper and must be replaced by the "holomorphic" extension introduced by the authors in [6].

1.

Let D be a pseudoconvex domain of $\mathbb{C}^n$ defined by $r < 0$ with C^∞ boundary bD. We use the standard notations $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ for the complex Laplacian and $Q(u, u) = ||\bar{\partial}u||^2 + ||\bar{\partial}^*u||^2$ for the energy form and some variants as, for an operator Op, $Q_{\mathrm{Op}}(u, u) = ||\mathrm{Op}\bar{\partial}u||^2 + ||\mathrm{Op}\bar{\partial}^*u||^2$. Here u is a antiholomorphic form of degree $k \le n - 1$ belonging to $D_{\bar{\partial}^*}$. We similarly define the tangential version of these objects, that is, $\square_b$, $\bar{\partial}_b$, $\bar{\partial}_b^*$, Q_{Op}^b. We take local coordinates (x, r) in $\mathbb{C}^n$ with $x \in \mathbb{R}^{2n-1}$ being the tangential coordinates and r, the equation of bD, serving as the last coordinate. We define the tangential s-Sobolev norm by $|||u|||_s := ||\Lambda^s u||_0$ where Λ^s is the standard tangential pseudodifferential operator with symbol $\Lambda^s_\xi = (1 + |\xi|^2)^{\frac{s}{2}}$. We note that

$$(1.1) \quad \begin{cases} ||\bar{\partial}u||^2_s + ||\bar{\partial}^*u||^2_s = \sum_{j \le s} Q_{\Lambda^{s-j}\partial_r^j}(u, u), \\ |||\bar{\partial}u|||^2_s + |||\bar{\partial}^*u|||^2_s = Q_{\Lambda^s}(u, u), \\ ||\bar{\partial}_b u_b||^2_s + ||\bar{\partial}_b^* u_b||^2_s = Q_{\Lambda^s}^b(u_b, u_b). \end{cases}$$

We decompose u in tangential and normal component, that is

$$u = u^\tau + u^\nu,$$

and further decompose in microlocal components (cf. [8])

$$u^\tau = u^{\tau+} + u^{\tau-} + u^{\tau 0}.$$

We similarly decompose $u_b = u_b^+ + u_b^- + u_b^0$. We use the notation $\bar{L}_n$ for the "normal" $(0, 1)$-vector field and $\bar{L}_1, ..., \bar{L}_{n-1}$ for the tangential

ones. We have therefore the description for the totally real tangential, resp. normal, vector field T, resp. ∂_r:

$$\begin{cases} T = i(L_n - \bar{L}_n), \\ \partial_r = L_n + \bar{L}_n. \end{cases}$$

From this, we get back $\bar{L}_n = \frac{1}{2}(\partial_r + iT)$. We denote by σ the symbol of a (pseudo)differential operator and by $\tilde{u}$ the partial tangential Fourier transform of u. We define a "holomorphic" extension $u^{\tau+(H)}$ by

$$(1.2) \qquad u^{\tau+(H)} = (2\pi)^{-2n+1} \int_{\mathbb{R}^{2n-1}} e^{ix\xi} e^{r\sigma(T)} \psi^+(\xi) \tilde{u}(\xi, 0) d\xi.$$

This definition has been introduced in [6]. Note that $\sigma(T) \underset{\sim}{>} (1+|\xi|^2)^{\frac{1}{2}}$ for ξ in $\operatorname{supp} \psi^+$ and (x, r) in a local patch; thus in the integral, the exponential is dominated by $e^{-|r|(1+|\xi|^2)^{\frac{1}{2}}}$ for $r < 0$. Differently from the harmonic extension by Kohn, the present one is well defined only in positive microlocalization. We can think of $u^{\tau+(H)}$ in two different ways: either as a modification of $u^{\tau+}$ or as an extension of u_b^+. We have a first relation from [8] p. 241, between a trace v_b and a general extension v: for any ϵ and suitable c_ϵ

$$(1.3) \qquad ||v_b|| \underset{\sim}{<} c_\epsilon |||v|||_{\frac{1}{2}} + \epsilon |||\partial_r v|||_{\frac{1}{2}}.$$

This can been seen in [8] p. 241 and [6] as for the small/large constant argument. As a specific property of our extension we have the reciprocal relation to (1.3), that is

$$(1.4) \qquad ||r^k u^{\tau+(H)}|| \underset{\sim}{<} ||u_b^+||_{-k-\frac{1}{2}}.$$

This is readily checked ([6] (1.12)). We denote by $\bar{\partial}^\tau$ the extension of $\bar{\partial}_b$ from $b\Omega$ to Ω which stays tangential to the level surfaces $r \equiv \operatorname{const}$. It acts on tangential forms u^τ and its action is $\bar{\partial}^\tau u^\tau = (\bar{\partial} u^\tau)^\tau$. We denote by $\bar{\partial}^{\tau*}$ its adjoint; thus $\bar{\partial}^{\tau*} u^\tau = \bar{\partial}^*(u^\tau)$. We use the notations $\Box^\tau$ and Q^τ for the corresponding Laplacian and energy form. We notice that

$$(1.5) \qquad \begin{aligned} Q(u^{\tau+(H)}, u^{\tau+(H)}) &= Q^\tau(u^{\tau+(H)}, u^{\tau+(H)}) + ||\bar{L}_n u^{\tau+(H)}||_0^2 \\ &= Q^\tau(u^{\tau+(H)}, u^{\tau+(H)}). \end{aligned}$$

We have to describe how (1.3) and (1.4) are affected by $\bar{\partial}$ and $\bar{\partial}^*$.

Proposition 1.1. *We have for any extension* v *of* v_b

$$(1.6) \qquad Q^b(v_b, v_b) \underset{\sim}{<} Q^\tau_{\Lambda^{\frac{1}{2}}}(v, v) + Q^\tau_{\partial_r \Lambda^{-\frac{1}{2}}}(v, v),$$

and, specifically for $u^{\tau+(H)}$

(1.7)
$$\begin{cases} Q^{\tau}(u^{\tau+(H)}, u^{\tau+(H)}) \underset{\sim}{\lesssim} Q^b_{\Lambda^{\frac{1}{2}}}(u_b^+, u_b^+) + ||u_b^+||^2_{-\frac{1}{2}} \\ \bar{L}_n u^{\tau+(H)} \equiv 0. \end{cases}$$

Proof. We have

$$\bar{\partial}^{\tau} v|_{bD} = \bar{\partial}_b v_b, \qquad \bar{\partial}^{\tau*} v|_{bD} = \bar{\partial}_b^* v_b.$$

Then, (1.6) follows from (1.3).

We pass to prove (1.7). We have $\bar{\partial}^{\tau} = \bar{\partial}_b + r\text{Tan}$, $\bar{\partial}^{\tau*} = \bar{\partial}_b^* + r\text{Tan}$ which yields

(1.8)
$$\begin{cases} \bar{\partial}^{\tau} u^{\tau+(H)} = (\bar{\partial}_b u_b)^{\tau+(H)} + r\text{Tan}\, u^{\tau+(H)}, \\ \bar{\partial}^{\tau*} u^{\tau+(H)} = (\bar{\partial}_b^* u_b)^{\tau+(H)} + r\text{Tan}\, u^{\tau+(H)}. \end{cases}$$

Application of (1.4) yields

$$||\bar{\partial}^{\tau} u^{\tau+(H)}||^2 + ||\bar{\partial}^{\tau*} u^{\tau+(H)}||^2 = ||(\bar{\partial}_b u_b)^{\tau+(H)}||^2 + ||(\bar{\partial}_b^* u_b)^{\tau+(H)}||^2 + ||r\text{Tan}\, u^{\tau+(H)}||^2$$
$$\underset{\sim}{\lesssim} ||\bar{\partial}_b u_b^+||^2_{-\frac{1}{2}} + ||\bar{\partial}_b^* u_b^+||^2_{-\frac{1}{2}} + ||u_b^+||^2_{-\frac{1}{2}},$$

which is the first of (1.7). The second is an easy consequence of the relation $\bar{L}_n = \frac{1}{2}(\partial_r + iT)$.

□

We finally decompose $u^{\tau+} = u^{\tau+(H)} + u^{\tau+(0)}$ which also serves as a definition of $u^{\tau+(0)}$.

Proposition 1.2. *Each of the forms $u^{\#} = u^{\nu}$, $u^{\tau-}$, $u^{\tau 0}$, $u^{\tau+(0)}$, u_b^-, u_b^0 enjoys elliptic estimates, that is*

(1.9) $\qquad ||\zeta u^{\#}||_s \underset{\sim}{\lesssim} ||\zeta' \bar{\partial} u^{\#}||_{s-1} + ||\zeta' \bar{\partial}^* u^{\#}||_{s-1} + ||u^{\#}||_0 \qquad s \geq 2.$

Proof. Estimate (1.9) follows, by iteration, from

(1.10) $\qquad ||\zeta u^{\#}||_s \underset{\sim}{\lesssim} ||\zeta \bar{\partial} u^{\#}||_{s-1} + ||\zeta \bar{\partial}^* u^{\#}||_{s-1} + ||\zeta' u^{\#}||_{s-1}.$

As for u^{ν} and $u^{\tau+(0)}$ this latter follows from $u^{\nu}|_{bD} \equiv 0$ and $u^{\tau+(0)}|_{bD} \equiv 0$. For the terms with $-$ and 0, this follows from the fact that $|\xi_T| \underset{\sim}{\lesssim} |\sigma(\bar{\partial})|$ in the region of 0-micolocalization and from $\sigma[\bar{\partial}, \bar{\partial}^*] \leq 0$ and $\sigma(T) < 0$ in the negative microlocalization. We refer to [2] formula (1) of Main theorem as a general reference but also give an outline of the proof. We start from

(1.11) $\qquad |||\zeta u^{\#}|||^2_1 \underset{\sim}{\lesssim} Q(\zeta u^{\#}, \zeta u^{\#}) + ||\zeta' u^{\#}||^2_0;$

this is the basic estimate for u^{ν} and $u^{\tau+(0)}$ (which vanish at bD) whereas it is [8] Lemma 8.6 for $u^{\tau-}$, $u^{\tau 0}$ and u_b^-, u_b^0. Applying (1.11)

to $\zeta\Lambda^{s-1}\zeta u^{\#}$ one gets the estimate of tangential norms for any s, that is, (1.10) with the usual norm replaced by the "triplet" norm. Finally, by non-characteristicity of $(\bar{\partial}, \bar{\partial}^*)$ one passes from tangential to full norms along the guidelines of [12] Theorem 1.9.7. The version of this argument for $\square$ can be found in [8] second part of p. 245.

$\square$

Let ζ and ζ' be a couple of cut-off with $\zeta \prec \zeta'$ in the sense that $\zeta'|_{\mathrm{supp}\,\zeta} \equiv 1$, and let s and l be a pair of indices.

Theorem 1.3. *The following two estimates are equivalent:*

(1.12)
$$||\zeta u_b||_s \underset{\sim}{<} ||\zeta'\bar{\partial}_b u_b||_{s+l} + ||\zeta'\bar{\partial}_b^* u_b||_{s+l} + ||u_b||_0 \quad \textit{for any } u_b \in C_c^\infty(b\Omega \cap U),$$

(1.13)
$$||\zeta u||_s \underset{\sim}{<} ||\zeta'\bar{\partial} u||_{s+l} + ||\zeta'\bar{\partial}^* u||_{s+l} + ||u||_0 \quad \textit{for any } u \in D_{\bar{\partial}^*} \cap C_c^\infty(\bar{\Omega} \cap U).$$

Remark 1.4. The above estimates (1.12) and (1.13) for any s, ζ, ζ' and for suitable l, characterize the local hypoellipticity of the system $(\bar{\partial}_b, \bar{\partial}_b^*)$ and $(\bar{\partial}, \bar{\partial}^*)$ respectively (cf. [9]). When $l > 0$, one says that the system has a "loss" of l derivatives; when $l < 0$, one says that it has a "gain" of $-l$ derivatives.

Proof. Because of Proposition 1.2, it suffices to prove (1.12) for u_b^+ and (1.13) for $u^{\tau +}$. It is also obviuos that we can consider cut-off functions ζ and ζ' in the only tangential coordinates, not in r. We start by proving that (1.12) implies (1.13). We recall the decomposition $u^{\tau +} = u^{\tau +(H)} + u^{\tau +(0)}$ and begin by estimating $u^{\tau +(H)}$. We have

(1.14)
$$|||\zeta u^{\tau +(H)}|||_s^2 \underset{\underset{(1.4)}{\sim}}{<} ||\zeta u_b^+||_{s-\frac{1}{2}}^2$$

$$\underset{\underset{(1.12)}{\sim}}{<} Q_{\Lambda^{s+l-\frac{1}{2}}\zeta'}^b(u_b^+, u_b^+) + ||u_b^+||_{-\frac{1}{2}}^2$$

$$\underset{\underset{(1.6)}{\sim}}{<} Q_{\Lambda^{s+l}\zeta'}^\tau(u^{\tau +}, u^{\tau +}) + Q_{\partial_r \Lambda^{s+l-\frac{1}{2}}\zeta'}^\tau(u^{\tau +}, u^{\tau +}) + ||u^{\tau +}||_0^2.$$

It remains to estimate $u^{\tau+(0)}$; since $u^{\tau+(0)}|_{bD} \equiv 0$, then by 1-elliptic estimates

(1.15)
$$|||\zeta u^{\tau+(0)}|||_s \underset{\sim}{<} Q_{\Lambda^{s-1}\zeta}(u^{\tau+(0)}, u^{\tau+(0)}) + |||\zeta' u^{\tau+(0)}|||^2_{s-1}$$
$$\underset{(1.10)}{}$$
$$\underset{\sim}{<} Q_{\Lambda^{s-1}\zeta}(u^{\tau+}, u^{\tau+}) + Q^\tau_{\Lambda^{s-1}\zeta}(u^{\tau+(H)}, u^{\tau+(H)}) + |||\zeta' u^{\tau+(0)}|||^2_{s-1}$$
$$\underset{\sim}{<} Q_{\Lambda^{s-1}\zeta}(u^{\tau+}, u^{\tau+}) + |||\zeta u^{\tau+(H)}|||^2_s + |||\zeta' u^{\tau+(H)}|||^2_{s-1} + |||\zeta' u^{\tau+(0)}|||^2_{s-1},$$

where we have used that $Q = Q^\tau$ over $u^{\tau+(H)}$ in the second inequality together with the estimate $Q^\tau_{\Lambda^{s-1}} \underset{\sim}{<} \Lambda^s$ in the third. We estimate terms in the last line. First, the term $|||\zeta u^{\tau+(H)}|||^2_s$ is estimated by means of (1.14). Next, the terms in $(s-1)$-norm can be brought to 0-norm by combined inductive use of (1.14) and (1.15) and eventually their sum is controlled by $||u^{\tau+}||^2_0$. We put together (1.14) and (1.15) (with the above further reductions), recall the first of (1.1) in order to estimate $Q^\tau \Lambda^s \zeta' + Q^\tau_{\partial_r \Lambda^{s-1}\zeta'}$ in the right of (1.14) and end up with

(1.16) $$|||\zeta u^{\tau+}|||_s \underset{\sim}{<} ||\zeta' \bar{\partial} u^{\tau+}||_s + ||\zeta' \bar{\partial}^* u^{\tau+}||_s + ||u^{\tau+}||_0.$$

Finally, by non-characteristicity of $(\bar{\partial}, \bar{\partial}^*)$ one passes from tangential to full norms in the left side of (1.16) along the guidelines of [12] Theorem 1.9.7. The version of this argument for $\square$ can be found in [8] second part of p. 245. Thus we get (1.13).

We prove the converse. Thanks to $\partial_r = \bar{L}_n + \text{Tan}$ and to the second of (1.7), we have $\partial_r u^{\tau+(H)} = \text{Tan}\, u^{\tau+(H)}$. It follows

(1.17)
$$||\zeta u^+_b||^2_s \underset{\sim}{<} c_\epsilon |||\zeta u^{\tau+(H)}|||^2_{s+\frac{1}{2}} + \epsilon |||\partial_r \zeta u^{\tau+(H)}|||^2_{s-\frac{1}{2}}$$
$$\underset{(1.3)}{}$$
$$\underset{\sim}{<} Q^\tau_{\Lambda^{s+l+\frac{1}{2}}\zeta'}(u^{\tau+(H)}, u^{\tau+(H)}) + \epsilon |||\zeta u^{\tau+(H)}|||^2_{s+\frac{1}{2}}$$
$$\underset{(1.13)}{}$$
$$\underset{\sim}{<} Q^b_{\Lambda^{s+l}\zeta'}(u^+_b, u^+_b) + \epsilon ||\zeta u^+_b||^2_s.$$
$$\underset{(1.7)}{}$$

We absorb the term with ϵ and get (1.12).

$\square$

Let N and G be the Neumann and Green operators, that is, the H^0-inverse of $\square$ in D and $\square_b$ in bD respectively.

Remark 1.5. (1.12) and (1.13) imply local regularity, but not exact s-Sobolev regularity, of G and N respectively. We first prove for N. We

start from remarking that

(1.18)
$$\begin{cases} \bar{\partial}^* N \text{ is exactly regular over } \mathrm{Ker}\bar{\partial}, \\ \bar{\partial} N \text{ is exactly regular over } \mathrm{Ker}\bar{\partial}^*. \end{cases}$$

As for the first, we put $u = \bar{\partial}^* N f$ for $f \in \mathrm{Ker}\,\bar{\partial}$. We have $(\bar{\partial}u = f, \bar{\partial}^* u = 0)$ and hence by (1.13) $||\zeta u||_s \underset{\sim}{<} ||\zeta' f||_s + ||u||_0$. To prove the second, we have just to put $u = \bar{\partial} N f$ for $f \in \mathrm{Ker}\,\bar{\partial}^*$ and reason likewise. It follows from (1.18), that the Bergman projection B is also regular. (Notice that exact regularity is perhaps lost by taking the additional $\bar{\partial}$ in $B := \mathrm{Id} - \bar{\partial}^* N \bar{\partial}$.) Finally, we exploit formula (5.36) in [11] in unweighted norms, that is, for $t = 0$:

(1.19)
$$\begin{aligned} N_q &= B_q(N_q\bar{\partial})(Id - B_{q-1})(\bar{\partial}^* N_q)B_q \\ &\quad + (Id - B_q)(\bar{\partial}^* N_{q+1})B_{q+1}(N_{q+1}\bar{\partial})(Id - B_q). \end{aligned}$$

Now, in the right side, the $\bar{\partial}N$'s and $\bar{\partial}^* N$'s are evaluated over $\mathrm{Ker}\bar{\partial}^*$ and $\mathrm{Ker}\,\bar{\partial}$ respectively; thus they are exactly regular. The B's are also regular and therefore such is N. This concludes the proof of the regularity of N. The proof of the regularity of G is similar, apart from replacing (1.19) by its version for the Green operator G stated in Section 5 of [4].

$\square$

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